Numerical Simulation of Steady-State Analysis to Predict Population Size over Time Using Linear Differential Equation Model

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Abstract

This study examines how to determine the steady-state solution (condition of a system that does not change over time) of a linear differential equation. It further explains the methods of solving a linear differential equation using integrating factor which produce the same general solution as Separation of Variable with the applications in the field of science and engineering. The numerical Simulations for table 1-3 were obtained analytically to predict the population size over a long period of time through the theory which states that over a long period of time, as the independent variable $t \to \infty$*, the population size will saturate and approach the steady–state value. The results obtained in these are fully presented and discussed.*

Keywords – *Steady-state analysis, population size, Integrating factor, particular solution, prediction.*

Introduction

The differential equation has many interesting applications in the field of science and engineering. Despite the wild applications of a differential equation, it is not common to find research papers which capture the full concepts of steady-state solution of a differential equation by using the method of integrating factor. It is possible to study the convergent of the general solution as the independent variable t approaches infinity. The key question is over a very long period of time, will the general solution approach the steady-state solution? This idea is the key background that must be satisfied before the population size can be estimated.

According to Ekaka-a (2009) investigated the computational and mathematical modeling of plant species in harsh climate. Nwagrabe etal [2] investigated the biodiversity effects and random disturbance on the ecosystem between two interacting yeast species using MATLAB ODE45 numerical scheme without the formulation of differential equation. Ekaka-a etal [3] studied the parametric sensitivity analysis of a dynamical system of continuous non-linear first order differential equations using the 1-norm 2-norm and infinity norm estimation. George etal [4] investigated the behaviour of a dynamical system using MATLAB ODE45 as an alternative method of verifying qualitatively the concept of stability of a unique positive steady-state solution and how the changes in initial data affects the stability of the steady state solution. Akpodee and Ekaka-a [5] studied the deterministic stability analysis of non linear first order differential equation using a numerical approach. Thieme [6] investigated the uniform persistence and permanence for non-autonomous semi-flows in population biology. Yan etal [7] worked on the stabilization and prediction of population system using a mathematical model. Luo [8] studied the population modeling by differential equations and predicting the extinction of antelopes in china using exponential and logistic model. This work extend the work of [6, 8] in the study of population biology using numerical approach. The purpose of this study is to analyze the steady state solution of a linear differential equation and predicting the population size over a long period of time (years) as it converges to a unique value which corresponds with the Steady-state value.

Formulation of Problem

Let p be the population of a place. Since P is a function of time, $P(t)$ is the given population with respect to time. We take r to represents the growth rate and Q the carrying capacity.

Hence, we obtain this differential equation model.

$$\frac{dP}{dt} = r(Q - P),\ r < 0 \qquad (1)$$

With the following initial conditions p(0)=15, p(0)=18 and p(0)=30.

where P =Population
r = Growth rate
Q = Carrying capacity
t = Time

With the initial conditions p(0)=15, p(0)=18 and p(0)=30, $r = 0.05$ and $Q = 1800$

Expanding the right hand side of equation (1), we obtain a linear differential equation model.

$$\frac{dP}{dt} + rP = rQ \qquad (2)$$

Let $K = rQ$ (3)

Substituting equation (3) into (2).

$$\frac{d}{dt} + rP = K \qquad (4)$$

Steady- State Solution

To determine the steady state solution, the change in dependent and independent variable is equal to zero using equation (4)

$$\frac{dP}{dt} = 0$$

$$rP = K$$

Let $P = P_e$ so that P_e will be different from the dependent variable.

$$rP_e = K$$

$$P_e = \frac{K}{r}$$

but $K = rQ$ from equation (3). Then substituting K into the expression

$$P_e = \frac{rQ}{r}$$

$$P_e = Q$$

$$\text{i.e. } P_e = 1800 \qquad (5)$$

Having known the steady state solution of this model, we conclude that the steady-state solution exist and then solve the linear differential equation model from equation (4) using the method integrating factor.

$$I.F = e^{a \int dt} \qquad (6)$$

Here the integrating factor is the coefficient of P

$$I.F = e^{r \int dt}$$

$$I.F = e^{rt}$$

Substituting e^{rt} into equation (4)

$$e^{rt}\frac{dP}{dt} + rPe^{rt} = Ke^{rt}$$

$$\frac{d}{dt}(Pe^{rt}) = Ke^{rt}$$

Integrating both side

$$e^{rt}P = K\int e^{rt}\,dt$$

$$e^{rt}P = \frac{K}{r}e^{rt} + C$$

$$P(t) = \frac{K}{r} + Ce^{-rt} \qquad (7)$$

since $K = rQ$, equation (7) becomes

$$P(t) = \frac{rQ}{r} + Ce^{-rt}$$

$$P(t) = Q + Ce^{-rt}$$

Where $Q = 1800, r = 0.05$

$$P(t) = 1800 + Ce^{-0.05t} \qquad (8)$$

Equation (7) is the general solution for this linear differential equation model.

Now, from the general solution obtained above, we take the limit as t →∞ such that

$$\underset{t\to\infty}{Limit}\ P(t) = \underset{t\to\infty}{Limit}\ 1800 + \underset{t\to\infty}{Limit}\ \frac{C}{e^{0.05(\infty)}} = 1800$$

Which corresponds with the steady-state solution in equation 6 and it is also called a **limiting value.**

Applying the initial condition for $P(0) = 15, P(0) = 18$ and $P(0) = 30, r = 0.05, Q = 1800$ into the general solution we obtain the following particular solutions.

$$P(t) = 1800 - 1785e^{-0.05t} \quad (9)$$

$$P(t) = 1800 - 1782e^{-0.05t} \quad (10)$$

$$P(t) = 1800 - 1770e^{-0.05t} \quad (11)$$

Predicting the Population Size

We shall predict P(t) as the parameter **t** varies at 50, 150, 250, 300, 350, 400, 450 & 500 for each of the particular solution obtained from the given initial conditions for equation 9-11.

Results and Discussion

$P(t) = 1800 - 1785e^{-0.05t}$

Varying t and keeping k constant, we tabulate below:

Example	t(years)	Constant	$Ke^{-0.05(t)}$ **K = 1770**	P(t)
1	50	1800	146.5217225	1,653.478277
2	100	1800	12.02723539	1,787.972765
3	150	1800	0.9872556	1799.012744
4	200	1800	0.081038874	1799.918961
5	250	1800	0.006652075912	1799.993348
6	300	1800	0.0005460356421	1799.999454
7	350	1800	0.00004482133493	1799.999955
8	400	1800	0.000003679159216	1799.999996
9	450	1800	0.0000003020037792	1800.000000
10	500	1800	0.00000024789978	1800.000000

Table 1: **showing the convergent of population size to the steady-state value over a long period at t=50, 100, 150, 200, 250, 300, 350, 400, 450 and 500 years.**

Table 1: Determines the stability of the given population and predicts the exponential growth of the population which converges to the steady–state after 200 years and fully saturate after 450 years as P(t)→ ∞. Hence, the steady state solution of this population exists.

Using the initial condition for p(0) = 18, the particular solution gives

$P(t) = 1800 - 1782e^{-0.05t}$

Varying t and keeping k constant, we tabulate below:

Example	t(years)	Constant	$Ke^{-0.05(t)}$ **K = 1,782**	P(t)
1	50	1800	146.2754675	1,653.724532
2	100	1800	12.00702155	1,787.992978
3	150	1800	0.985596347	1,799.014404
4	200	1800	0.080902674	1,799.919097
5	250	1800	0.006640895953	1,799.993359
6	300	1800	0.0005451179351	1,799.999455
7	350	1800	0.000003672975755	1,799.999955
8	400	1800	0.000003672975755	1,799.999996

9	450	1800	0.0000003014962098	1800.000000
10	500	1800	0.00000002474831597	1800.000000

Table 2: showing the convergent of population size to the steady-state value over a long period at t=50, 100, 150, 200, 250, 300, 350, 400, 450 and 500 years.

The analysis of Table 2: Show the stability of the given population and predicts the exponential growth of the population which converges to the steady–state after 100 years and fully saturate after 450 years as P(t)→ ∞.

Using the initial condition for P(0)=30 which gives the particular solution
$P(t) = 1800 - 1770e^{-0.05t}$

Varying t and keeping k constant, we tabulate below:

Example	t(years)	Constant	$Ke^{-0.05(t)}$ **K = 1,7 70**	P(t)
1	50	1800	145.2904476	1.654.709552
2	100	1800	11.92616619	1,788.073834
3	150	1800	0.978959335	1,799.021041
4	200	1800	0.080357875	1,799.919642
5	250	1800	0.006596176115	1,799.993404
6	300	1800	0.0005414471075	1,799.999459
7	350	1800	0.000004444468506	1,799.999956
8	400	1800	0.000003648241912	1,799.999956
9	450	1800	0.0000002994659323	1800.000000
10	500	1800	0.00000002458166064	1800.000000

Table 3: showing the convergent of population size to the steady-state value over a long period at t=50, 100, 150, 200, 250, 300, 350, 400, 450 and 500 years.

The analysis of Table 3: Show the stability of the given population and predicts the exponential growth of the population which converges to the steady–state after 100 years and fully saturate after 450 years as P(t) → ∞. This further explains that the population of this system does not change over time.

Conclusion

The significant findings of Numerical simulation of steady-state analysis and predicting the population size over time are as follows:

1. The stead- state value is the same as the carrying capacity of a given population which is the maximum growth of the population.
2. The existence of the steady-state solution determines the stability of population, otherwise it's either unstable or grow into extinction.
3. The population growth converges faster as the initial conditions increases.
4. If we take the limit as the independent variable t→∞, we will have a unique convergent that corresponds with steady-state value.
5. We must apply the initial condition in order to obtain the particular solution from the general solution in order to predict the population size.

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